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On A Class Between Devaney Chaotic and Li–Yorke Chaotic Generalized Shift Dynamical Systems

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Abstract

In the following text, for finite discrete X with at least two elements, nonempty countable Γ, and $\varphi : \Gamma \to \Gamma$ we prove the generalized shift dynamical system $(X^\Gamma, \sigma_\varphi)$ is densely chaotic if and only if $\varphi : \Gamma \to \Gamma$ does not have any (quasi–)periodic point. Hence the class of all densely chaotic generalized shifts on X^Γ is intermediate between the class of all Devaney chaotic generalized shifts on X^Γ and the class of all Li–Yorke chaotic generalized shifts on X^Γ. In addition, these inclusions are proper for infinite countable Γ. Moreover we prove $(X^\Gamma, \sigma_\varphi)$ is Li–Yorke sensitive (resp. sensitive, strongly sensitive, asymptotic sensitive, syndetically sensitive, cofinitely sensitive, multi–sensitive, ergodically sensitive, spatiotemporally chaotic, Li–Yorke chaotic) if and only if $\varphi : \Gamma \to \Gamma$ has at least one non–quasi–periodic point.

Keywords: asymptotic sensitive; densely chaotic; Li-Yorke sensitive; spatiotemporally chaotic; strongly sensitive; generalized shift.

1 Introduction

One of the first concepts is introduced in the intermediate mathematical level, is the concept of a self-map $\varphi : \Gamma \to \Gamma$. We recall that a point $a \in \Gamma$ is a *periodic point* of $\varphi : \Gamma \to \Gamma$ if there exists $n \geq 1$ with $\varphi^n(a) = a$ and it is a *quasi–periodic point* of $\varphi : \Gamma \to \Gamma$ if there exist $n > m \geq 1$ with $\varphi^n(a) = \varphi^m(a)$. We denote the collection of all periodic (resp. non–quasi–periodic) points of $\varphi : \Gamma \to \Gamma$ with $Per(\varphi)$ (resp. $W(\varphi)$).

One may find with an elementary approach that the map $\varphi : \Gamma \to \Gamma$ does not have any periodic point if and only if $W(\varphi) = \Gamma$ (note that if $\alpha \in \Gamma \setminus W(\varphi)$, then it is a quasi–periodic point and there exist $n > m \geq 1$ with $\varphi^m(\alpha) = \varphi^n(\alpha)$ by $\varphi^{n-m}(\varphi^m(\alpha)) = \varphi^n(\alpha)$ we have $\varphi^{n-m}(\varphi^m(\alpha)) = \varphi^m(\alpha)$ then $\varphi^m(\alpha) \in Per(\varphi)$, and $Per(\varphi) \neq \varnothing$).

Now for nonempty set Γ and $\varphi : \Gamma \to \Gamma$ we have the following diagram:

$$\varphi \text{ is one to one and } Per(\varphi) = \varnothing \Longrightarrow W(\varphi) = \Gamma \Longrightarrow W(\varphi) \neq \varnothing.$$

For "suitable" generalized shift dynamical system $(X^\Gamma, \sigma_\varphi)$ which we deal in this paper, the first statement in the above diagram "φ is one to one and $Per(\varphi) = \varnothing$" is equivalent to "$(X^\Gamma, \sigma_\varphi)$ is Devaney chaotic" [18, Theorem 2.13] and the last statement "$W(\varphi) \neq \varnothing$" is equivalent to "$(X^\Gamma, \sigma_\varphi)$ is Li-Yorke chaotic" [17, Theorem 3.3]. We return to this diagram in Section 4 and show that the second statement "$W(\varphi) = \Gamma$" is equivalent to "$(X^\Gamma, \sigma_\varphi)$ is densely chaotic", moreover the implications of the diagram are not reversible for infinite Γ.

Moreover "sensitive to initial conditions" or sometimes known as "butterfly effect" may is the first concept in sensitivity approach for a large group of the mathematicians (see [3]). However, regarding different dynamical points of view nowadays we can find various types of sensitivity, old ones and new ones in metric dynamical systems like mean sensitive, Li–Yorke sensitivity, strongly sensitive, ergodically sensitive, multi–sensitive, cofinitely sensitive ... (see e.g., [5, 7, 8, 9, 14, 21]), or even in general dynamical systems [4].

In last section of this text, a collection of different types of sensitivities in the category of compact metric generalized shift dynamical systems are compared.

In this text by $\mathbb{N}$, we meant the set of positive integers $\{1, 2, \ldots\}$. Also for finite set A, $|A|$ denotes the cardinality of A.

What is a Generalized Shift?

By a *dynamical system* (or briefly *system*) (Z, f) we mean a topological space Z and continuous map $f : Z \to Z$. One of the most famous dynamical systems is one-sided shift dynamical system $\sigma : \{1, \ldots, k\}^{\mathbb{N}} \to \{1, \ldots, k\}^{\mathbb{N}}$ with $\sigma((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$ (for $(x_n)_{n\in\mathbb{N}} \in \{1, \ldots, k\}^{\mathbb{N}}$). Studying dynamical and non-dynamical properties of one-sided shift has been considered by several authors, however the reader may find interesting ideas in [12] too.

For nonempty arbitrary set X with at least two elements and nonempty set Γ, we call $\sigma_\varphi : X^\Gamma \to X^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ (for $(x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$) a *generalized shift* (generalizing of an idea or concept is common in mathematics (see [6] and [10] as examples)). If X has a topological structure and X^Γ equipped with product (pointwise convergence) topology, then it is evident that $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is continuous, so we may consider the dynamical system $(X^\Gamma, \sigma_\varphi)$. Moreover X^Γ is a compact metrizable space if and only if X is compact metrizable and Γ is countable. Generalized shift has been introduced for the first time in [16], which has been followed by studying several properties of generalized shifts like, topological entropy [15], Devaney chaos [18] and Li-Yorke chaos [17, 13].

2 Preliminaries in Dynamical Systems

In the dynamical system (Z, f) with compact metric phase space (Z, d), we say $x, y \in Z$ are *scrambled* or (x, y) is a *scrambled pair* of (Z, f) if

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0 .$$

We denote the collection of all scarmbled pairs of (Z, f) with $S((Z, d), f)$ or briefly $S(Z, f)$. We call a subset A of Z with at least two elements an *scrambled set* if every distinct pairs of elements of A is an scrambled pair, i.e., $A \times A \subseteq S(Z, f) \cup \Delta_Z$, where $\Delta_Z = \{(z, z) : z \in Z\}$. Also for $\varepsilon > 0$ denote $\{(x, y) \in S(Z, f) : \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon\}$ by $S_\varepsilon((Z, d), f)$ or briefly $S_\varepsilon(Z, f)$.

Note 2.1. Suppose d and d' are two compatible metrics on compact metrizable space Z, then for every $\varepsilon > 0$ exists $\delta > 0$ such that

$$\forall x, y \in Z\ (d'(x, y) < \delta \Rightarrow d(x, y) < \varepsilon).$$

Now suppose $\varepsilon, \delta > 0$ satisfy the above statement, $f : Z \to Z$ is continuous, and $(x, y) \in S_\varepsilon((Z, d), f)$, then $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ thus there exists subsequence $\{d(f^{n_k}(x), f^{n_k}(y))\}_{k\geq 1}$ such that

$$\lim_{k\to\infty} d(f^{n_k}(x), f^{n_k}(y)) = \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 .$$

Since $Z \times Z$ is a compact metrizable space, there exists subsequence $\{(f^{n_{k_l}}(x), f^{n_{k_l}}(y))\}_{l\geq 1}$ of $\{(f^{n_k}(x), f^{n_k}(y))\}_{k\geq 1}$ converging to a point of $Z \times Z$ like (z, w), hence $d(z, w) = \lim_{l\to\infty} d(f^{n_{k_l}}(x), f^{n_{k_l}}(y)) = \lim_{k\to\infty} d(f^{n_k}(x), f^{n_k}(y)) = 0$ and $z = w$, therefore $\lim_{l\to\infty} d'(f^{n_{k_l}}(x), f^{n_{k_l}}(y)) = d'(z, w) = 0$, and

$$\liminf_{n\to\infty} d'(f^n(x), f^n(y)) = 0 .$$

Moreover, $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon$, hence there exists subsequence $\{d(f^{m_t}(x), f^{m_t}(y))\}_{t\geq 1}$ such that

$$\lim_{t\to\infty} d(f^{m_t}(x), f^{m_t}(y)) = \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon .$$

Again, since $Z \times Z$ is a compact metrizable space, there exists subsequence $\{(f^{m_{t_l}}(x), f^{m_{t_l}}(y))\}_{l\geq 1}$ of $\{(f^{m_t}(x), f^{m_t}(y))\}_{t\geq 1}$ converging to a point of $Z \times Z$ like (u, v), hence $d(u, v) = \lim_{l\to\infty} d(f^{m_{t_l}}(x), f^{m_{t_l}}(y)) = \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon$ therefore $d'(u, v) \geq \delta$. Thus

$$\limsup_{n\to\infty} d'(f^n(x), f^n(y)) \geq \lim_{l\to\infty} d'(f^{m_{t_l}}(x), f^{m_{t_l}}(y)) = d'(u, v) \geq \delta > \delta/2 ,$$

which shows $(x, y) \in S_{\delta/2}((Z, d'), f)$ So we have, $S_\varepsilon((Z, d), f) \subseteq S_{\delta/2}((Z, d'), f)$.

By using the above argument we have:

$$\forall \varepsilon > 0\, \exists \mu > 0\, (S_\varepsilon((Z, d), f) \subseteq S_\mu((Z, d'), f))$$

and $S((Z,d),f) = S((Z,d'),f)$, i.e. the definition of an scrambled pair is independent of chosen compatible metric on Z (for more details see [17]).

We recall that for $\varepsilon > 0$ the dynamical system (Z,f), with compact metric phase space (Z,d), is:

- *Li-Yorke chaotic*, if Z has an uncountable scrambled subset;
- *Li-Yorke sensitive*, if there exists $\kappa > 0$ such that for every $x \in Z$ and open neighbourhood U of x there exists $y \in U$ with $(x,y) \in S_\kappa(Z,f)$ [7];
- *densely $\varepsilon-$chaotic*, if $S_\varepsilon(Z,f)$ is a dense subset of $Z \times Z$ [11];
- *spatiotemporally chaotic*, if for every $x \in Z$ and open neighbourhood U of x there exists $y \in U$ such that x, y are scrambled [20];
- *densely chaotic*, if $S(Z,f)$ is a dense subset of $Z \times Z$ [11];
- *topological transitive*, if for all opene (nonempty and open) subsets U, V of Z there exists $n \geq 1$ with $U \cap f^n(V) \neq \varnothing$ [2];
- *Devaney chaotic*, if it is topological transitive, $Per(f)$ is dense in Z (i.e., (Z,f) has *dense periodic points*), and it is sensitive dependence to initial conditions (by [1] sensitivity dependence to initial conditions is redundant).

Convention 2.2. : In the following text suppose X is a finite discrete space with at least two elements, Γ is a nonempty countable set, and $\varphi : \Gamma \to \Gamma$ is a self-map.
Suppose $\Gamma = \{\beta_1, \beta_2, \ldots\}$ equip X^Γ with metric:

$$D((x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma}) = \sum_{n\geq 1} \frac{\delta(x_{\beta_n}, y_{\beta_n})}{2^n} \qquad ((x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma} \in X^\Gamma)$$

where

$$\delta(a,b) = \begin{cases} 0 & a = b\,, \\ 1 & a \neq b\,. \end{cases}$$

Then D is a compatible metric with product topology of X^Γ.

3 Densely Chaotic Generalized Shift Dynamical Systems

In this section we prove that the system $(X^\Gamma, \sigma_\varphi)$ is densely chaotic if and only if $\varphi : \Gamma \to \Gamma$ does not have any periodic point, i.e. $W(\varphi) = \Gamma$.

Lemma 3.1. *If $\varphi : \Gamma \to \Gamma$ does not have any periodic point, then for all finite nonempty subsets A, B of Γ, $\{n \in \mathbb{Z} : \varphi^n(A) \cap B \neq \varnothing\}$ has at most* $\mathrm{card}(A)\mathrm{card}(B)$ *elements and it is finite.*

Proof. Since $\varphi : \Gamma \to \Gamma$ does not have any periodic point, it does not have any quasi–periodic point too. We prove for $\alpha, \beta \in \Gamma$, $K = \{n \in \mathbb{Z} : \beta \in \varphi^n(\{\alpha\})\}$ is void or singleton. Otherwise there exists distinct $n, m \in K$. Suppose $n < m$, we have the following cases:

- $0 \leq n < m$. In this case $\varphi^{n+1}(\alpha) = \varphi(\beta) = \varphi^{m+1}(\alpha)$ and α is a quasi–periodic point of φ.

- $n < 0 \leq m$. In this case $\varphi^{-n}(\beta) = \alpha$ and $\beta = \varphi^m(\alpha)$ which leads to $\varphi^{-n+m+1}(\alpha) = \varphi^{-n+1}(\beta) = \varphi(\alpha)$ and α is a quasi–periodic point of φ.
- $n < m < 0$. In this case $\varphi^{-n}(\beta) = \alpha = \varphi^{-m}(\beta)$ and β is a quasi–periodic point of φ.

By using the above cases $\varphi : \Gamma \to \Gamma$ has a quasi–periodic point, which is a contradiction. Hence K has at most one element, which leads to the fact that

$$\{n \in \mathbb{Z} : \varphi^n(A) \cap B \neq \varnothing\} = \bigcup \{\{n \in \mathbb{Z} : \beta \in \varphi^n(\{\alpha\})\} : (\alpha, \beta) \in A \times B\},$$

has at most $\mathrm{card}(A)\mathrm{card}(B)$ elements. □

Lemma 3.2. *Suppose $\varphi : \Gamma \to \Gamma$ does not have any periodic point, $((x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma}) \in S(X^\Gamma, \sigma_\varphi)$, $((z_\alpha)_{\alpha\in\Gamma}, (w_\alpha)_{\alpha\in\Gamma}) \in X^\Gamma \times X^\Gamma$ and there exist $\psi_1, \ldots, \psi_n \in \Gamma$ such that for all $\alpha \in \Gamma \setminus \{\psi_1, \ldots, \psi_n\}$ we have both $x_\alpha = z_\alpha$ and $y_\alpha = w_\alpha$. Then $((z_\alpha)_{\alpha\in\Gamma}, (w_\alpha)_{\alpha\in\Gamma}) \in S(X^\Gamma, \sigma_\varphi)$ and:*

$$\limsup_{t\to\infty} D(\sigma_\varphi^t((x_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^t((y_\alpha)_{\alpha\in\Gamma})) = \limsup_{t\to\infty} D(\sigma_\varphi^t((z_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^t((w_\alpha)_{\alpha\in\Gamma}))\,.$$

Proof. Given $\varepsilon > 0$ there exists $N \geq 1$ with $\sum\limits_{i\geq N} \frac{1}{2^i} < \frac{\varepsilon}{2}$. Using Lemma 3.1 there exists $K \geq 1$ with

$$\varphi^t(\{\beta_1, \ldots, \beta_N\}) \cap \{\psi_1, \ldots, \psi_n\} = \varnothing \text{ for all } t \geq K.$$

Thus $x_{\varphi^t(\beta_i)} = z_{\varphi^t(\beta_i)}$ and $y_{\varphi^t(\beta_i)} = w_{\varphi^t(\beta_i)}$ for all $1 \leq i \leq N$ and $t \geq K$. Hence for all $t \geq K$ we have:

$$\begin{aligned}
&\left|D(\sigma_\varphi^t((x_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^t((y_\alpha)_{\alpha\in\Gamma})) - D(\sigma_\varphi^t((z_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^t((w_\alpha)_{\alpha\in\Gamma}))\right| \\
&= \left|\sum_{i\geq 1} \frac{\delta(x_{\varphi^t(\beta_i)}, y_{\varphi^t(\beta_i)})}{2^i} - \sum_{i\geq 1} \frac{\delta(z_{\varphi^t(\beta_i)}, w_{\varphi^t(\beta_i)})}{2^i}\right| \\
&\leq \left|\sum_{1\leq i\leq N} \left(\frac{\delta(x_{\varphi^t(\beta_i)}, y_{\varphi^t(\beta_i)}) - \delta(z_{\varphi^t(\beta_i)}, w_{\varphi^t(\beta_i)})}{2^i}\right)\right| + \\
&\qquad \left|\sum_{i>N} \left(\frac{\delta(x_{\varphi^t(\beta_i)}, y_{\varphi^t(\beta_i)}) - \delta(z_{\varphi^t(\beta_i)}, w_{\varphi^t(\beta_i)})}{2^i}\right)\right| \\
&= \left|\sum_{1\leq i\leq N} \left(\frac{\delta(x_{\varphi^t(\beta_i)}, y_{\varphi^t(\beta_i)}) - \delta(x_{\varphi^t(\beta_i)}, y_{\varphi^t(\beta_i)})}{2^i}\right)\right| + \\
&\qquad \left|\sum_{i>N} \left(\frac{\delta(x_{\varphi^t(\beta_i)}, y_{\varphi^t(\beta_i)}) - \delta(z_{\varphi^t(\beta_i)}, w_{\varphi^t(\beta_i)})}{2^i}\right)\right| \\
&= \left|\sum_{i>N} \left(\frac{\delta(x_{\varphi^t(\beta_i)}, y_{\varphi^t(\beta_i)}) - \delta(z_{\varphi^t(\beta_i)}, w_{\varphi^t(\beta_i)})}{2^i}\right)\right| \\
&\leq \sum_{i>N} \left(\frac{\delta(x_{\varphi^t(\beta_i)}, y_{\varphi^t(\beta_i)}) + \delta(z_{\varphi^t(\beta_i)}, w_{\varphi^t(\beta_i)})}{2^i}\right) \leq \sum_{i>N} \frac{2}{2^i} < \varepsilon\,.
\end{aligned}$$

Using
$$\left|D(\sigma_\varphi^t((x_\alpha)_{\alpha\in\Gamma}),\sigma_\varphi^t((y_\alpha)_{\alpha\in\Gamma}))-D(\sigma_\varphi^t((z_\alpha)_{\alpha\in\Gamma}),\sigma_\varphi^t((w_\alpha)_{\alpha\in\Gamma}))\right|<\varepsilon \quad (\forall t\geq K)\,,$$
we have:
$$\left|\limsup_{t\to\infty} D(\sigma_\varphi^t((x_\alpha)_{\alpha\in\Gamma}),\sigma_\varphi^t((y_\alpha)_{\alpha\in\Gamma}))-\limsup_{t\to\infty} D(\sigma_\varphi^t((z_\alpha)_{\alpha\in\Gamma}),\sigma_\varphi^t((w_\alpha)_{\alpha\in\Gamma}))\right|\leq\varepsilon\,,$$
and
$$\left|\liminf_{t\to\infty} D(\sigma_\varphi^t((x_\alpha)_{\alpha\in\Gamma}),\sigma_\varphi^t((y_\alpha)_{\alpha\in\Gamma}))-\liminf_{t\to\infty} D(\sigma_\varphi^t((z_\alpha)_{\alpha\in\Gamma}),\sigma_\varphi^t((w_\alpha)_{\alpha\in\Gamma}))\right|\leq\varepsilon\,.$$
Note to the fact that $\varepsilon>0$ is arbitrary, we have:
$$\limsup_{t\to\infty} D(\sigma_\varphi^t((x_\alpha)_{\alpha\in\Gamma}),\sigma_\varphi^t((y_\alpha)_{\alpha\in\Gamma}))=\limsup_{t\to\infty} D(\sigma_\varphi^t((z_\alpha)_{\alpha\in\Gamma}),\sigma_\varphi^t((w_\alpha)_{\alpha\in\Gamma}))\,,$$
and
$$\liminf_{t\to\infty} D(\sigma_\varphi^t((x_\alpha)_{\alpha\in\Gamma}),\sigma_\varphi^t((y_\alpha)_{\alpha\in\Gamma}))=\liminf_{t\to\infty} D(\sigma_\varphi^t((z_\alpha)_{\alpha\in\Gamma}),\sigma_\varphi^t((w_\alpha)_{\alpha\in\Gamma}))\,,$$
which lead to the desired result. □

The following remark deal with the situation $W(\varphi)\neq\varnothing$, and its connection to Li–Yorke and topological chaoticity of $(X^\Gamma,\sigma_\varphi)$. Let's recall that the system (Z,f) is *topologically chaotic* if it has positive topological entropy (for the definition of topological entropy and more details see [19]).

Remark 3.1. *By [15, Theorem 4.7], topological entropy of $\sigma_\varphi: X^\Gamma\to X^\Gamma$ is equal to $\mathfrak{o}(\varphi)\log|X|$, where:*

$$\mathfrak{o}(\varphi)=\sup(\{0\}\cup\{n\in\mathbb{N}: \text{there exist } \alpha_1,\ldots,\alpha_n\in\Gamma \text{ such that } \{\varphi^m(\alpha_1)\}_{m\geq1},\ldots,\{\varphi^m(\alpha_n)\}_{m\geq1} \text{ are infinite and pairwise disjoint }\}).$$

Hence $(X^\Gamma,\sigma_\varphi)$ is topological chaotic if and only if $\mathfrak{o}(\varphi)>0$, i.e., $\varphi:\Gamma\to\Gamma$ has at least one non-quasi–periodic point. So by [17, Theorem 3.3] the system $(X^\Gamma,\sigma_\varphi)$ is Li-Yorke chaotic (resp. has an scrambled pair) if and only if the map $\varphi:\Gamma\to\Gamma$ has at least one non–quasi–periodic point which is equivalent to topological chaoticity of $(X^\Gamma,\sigma_\varphi)$ in its turn.

Lemma 3.3. *If $\varphi:\Gamma\to\Gamma$ does not have any periodic point, then there exists $\mu>0$ such that:*
$$\forall x\in X^\Gamma\ \exists y\in X^\Gamma\ \ ((x,y)\in S(X^\Gamma,\sigma_\varphi)\wedge\limsup_{t\to\infty} D(\sigma_\varphi^t(x),\sigma_\varphi^t(y))=\mu)\,.$$

Proof. Since $\varphi:\Gamma\to\Gamma$ does not have any periodic point, it does not have any quasi–periodic point too, and all of points of Γ are non–quasi–periodic, hence by Remark 3.1 there exists $((p_\alpha)_{\alpha\in\Gamma},(q_\alpha)_{\alpha\in\Gamma})\in S(X^\Gamma,\sigma_\varphi)$, let:
$$\mu:=\limsup_{t\to\infty} D(\sigma_\varphi^t((p_\alpha)_{\alpha\in\Gamma}),\sigma_\varphi^t((q_\alpha)_{\alpha\in\Gamma}))\,.$$
Consider $(x_\alpha)_{\alpha\in\Gamma}\in X^\Gamma$ and choose $(y_\alpha)_{\alpha\in\Gamma}\in X^\Gamma$ such that:
$$y_\alpha\begin{cases} =x_\alpha & p_\alpha=q_\alpha\,,\\ \in\{p_\alpha,q_\alpha\}\setminus\{x_\alpha\} & p_\alpha\neq q_\alpha\,.\end{cases}$$
Hence $x_\alpha=y_\alpha$ if and only if $p_\alpha=q_\alpha$, therefore
$$\delta(x_\alpha,y_\alpha)=\delta(p_\alpha,q_\alpha)\quad(\forall\alpha\in\Gamma)\,.$$

For all $t \geq 1$ we have:

$$D(\sigma_\varphi^t((x_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^t((y_\alpha)_{\alpha\in\Gamma})) = \sum_{i\geq 1} \frac{\delta(x_{\varphi^t(\beta_i)}, y_{\varphi^t(\beta_i)})}{2^i} = \sum_{i\geq 1} \frac{\delta(p_{\varphi^t(\beta_i)}, q_{\varphi^t(\beta_i)})}{2^i}$$
$$= D(\sigma_\varphi^t((p_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^t((q_\alpha)_{\alpha\in\Gamma}))$$

which leads to

$$\liminf_{t\to\infty} D(\sigma_\varphi^t((x_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^t((y_\alpha)_{\alpha\in\Gamma})) = \liminf_{t\to\infty} D(\sigma_\varphi^t((p_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^t((q_\alpha)_{\alpha\in\Gamma})) = 0\,,$$

$$\limsup_{t\to\infty} D(\sigma_\varphi^t((x_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^t((y_\alpha)_{\alpha\in\Gamma})) = \limsup_{t\to\infty} D(\sigma_\varphi^t((p_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^t((q_\alpha)_{\alpha\in\Gamma})) = \mu > 0\,,$$

and $((x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma}) \in S(X^\Gamma, \sigma_\varphi)$. □

Now we have the following lemma.

Lemma 3.4. *If $(X^\Gamma, \sigma_\varphi)$ is densely chaotic, then $W(\varphi) = \Gamma$ (i.e., $\varphi : \Gamma \to \Gamma$ does not have any periodic point).*

Proof. Consider $\beta \in Per(\varphi)$ and $n \geq 1$ with $\varphi^n(\beta) = \beta$, we prove $(X^\Gamma, \sigma_\varphi)$ is not densely chaotic. We may also suppose $\beta_1 = \beta, \beta_2 = \varphi(\beta), \dots, \beta_n = \varphi^{n-1}(\beta)$. Choose distinct $p, q \in X$ and let:

$$U_\alpha = \begin{cases} \{p\} & \alpha = \beta_1, \dots, \beta_n\,, \\ X & \text{otherwise}\,, \end{cases} \qquad V_\alpha = \begin{cases} \{q\} & \alpha = \beta_1, \dots, \beta_n\,, \\ X & \text{otherwise}\,, \end{cases}$$

also let:

$$U = \prod_{\alpha\in\Gamma} U_\alpha \quad , \quad V = \prod_{\alpha\in\Gamma} V_\alpha\,,$$

then $U \times V$ is an opene subset of $X^\Gamma \times X^\Gamma$. Consider $(x, y) = ((x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma}) \in U \times V$ we have $x_{\beta_1} = \cdots = x_{\beta_n} = p$ and $y_{\beta_1} = \cdots = y_{\beta_n} = q$. For all $k \geq 1$ we have $\varphi^k(\beta) \in \{\beta_1, \dots, \beta_n\}$ which leads to $x_{\varphi^k(\beta)} = p$ and $y_{\varphi^k(\beta)} = q$, thus:

$$D(\sigma_\varphi^k((x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma})) = D((x_{\varphi^k(\alpha)})_{\alpha\in\Gamma}, (y_{\varphi^k(\alpha)})_{\alpha\in\Gamma})$$
$$\geq \frac{1}{2}\delta(x_{\varphi^k(\beta)}, y_{\varphi^k(\beta)}) = \frac{1}{2}\delta(p, q) = \frac{1}{2}\,.$$

Hence

$$\liminf_{k\to\infty} D(\sigma_\varphi^k((x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma})) \geq \frac{1}{2}$$

and $(x, y) \notin S(X^\Gamma, \sigma_\varphi)$. Using $(U \times V) \cap S(X^\Gamma, \sigma_\varphi) = \varnothing$ we have the desired result. □

Now we are ready to characterize densely chaotic generalized shifts.

Theorem 3.1 (Densely chaotic generalized shifts)**.** *For finite discrete X with at least two elements, nonempty countable set Γ and $\varphi : \Gamma \to \Gamma$, the following statements are equivalent:*

1. *for some $\varepsilon > 0$, $(X^\Gamma, \sigma_\varphi)$ is densely $\varepsilon-$chaotic;*
2. *the system $(X^\Gamma, \sigma_\varphi)$ is densely chaotic;*
3. *the map $\varphi : \Gamma \to \Gamma$ does not have any periodic point (i.e., $W(\varphi) = \Gamma$).*

Proof. Clearly (1) implies (2). By Lemma 3.4, (2) implies (3).
($3 \Rightarrow 1$): Suppose $W(\varphi) = \Gamma$ by Lemma 3.3 there exists $\mu > 0$ such that for all $x = (x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$, there exists $y = (y_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$ with $((x_\alpha)_{\alpha\in\Gamma}, (y_\alpha)_{\alpha\in\Gamma}) \in S(X^\Gamma, \sigma_\varphi)$ and

$$\limsup_{t\to\infty} D(\sigma_\varphi^t((x_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^t((y_\alpha)_{\alpha\in\Gamma})) = \mu\,.$$

Suppose W is an opene subset of $X^\Gamma \times X^\Gamma$, there exist opene subsets U, V of X^Γ with $U \times V \subseteq W$. Choose $u = (u_\alpha)_{\alpha\in\Gamma} \in U, v = (v_\alpha)_{\alpha\in\Gamma} \in V$ and $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\prod\limits_{\alpha\in\Gamma} U_\alpha \subseteq U$, $\prod\limits_{\alpha\in\Gamma} V_\alpha \subseteq V$ with

$$U_\alpha = \begin{cases} \{u_\alpha\} & \alpha = \psi_1, \ldots, \psi_n\,, \\ X & \text{otherwise}\,, \end{cases} \quad \textit{and} \quad V_\alpha = \begin{cases} \{v_\alpha\} & \alpha = \psi_1, \ldots, \psi_n\,, \\ X & \text{otherwise}\,. \end{cases}$$

Then for:

$$z_\alpha = \begin{cases} u_\alpha & \alpha = \psi_1, \ldots, \psi_n\,, \\ x_\alpha & \text{otherwise}\,, \end{cases} \quad \textit{and} \quad w_\alpha = \begin{cases} v_\alpha & \alpha = \psi_1, \ldots, \psi_n\,, \\ y_\alpha & \text{otherwise}\,. \end{cases}$$

By Lemma 3.2, we have $(z, w) := ((z_\alpha)_{\alpha\in\Gamma}, (w_\alpha)_{\alpha\in\Gamma}) \in S(X^\Gamma, \sigma_\varphi)$ with

$$\limsup_{t\to\infty} D(\sigma_\varphi^t(z), \sigma_\varphi^t(w)) = \limsup_{t\to\infty} D(\sigma_\varphi^t(x), \sigma_\varphi^t(y)) = \mu\,.$$

Thus, $(z, w) \in (U \times V) \cap S_{\frac{\mu}{2}}(X^\Gamma, \sigma_\varphi)$ and $W \cap S_{\frac{\mu}{2}}(X^\Gamma, \sigma_\varphi) \neq \varnothing$ for each opene subset W of $X^\Gamma \times X^\Gamma$. Therefore, $(X^\Gamma, \sigma_\varphi)$ is $\frac{\mu}{2}-$densely chaotic. □

4 Sensitivity in Generalized Shift Dynamical Systems

We recall that the dynamical system (Z, f) with compact metric phase space (Z, d) is [14]:

- *sensitive* if there exists $\varepsilon > 0$ such that for all $x \in Z$ and open neighbourhood V of x there exist $n \geq 0$ and $y \in V$ with $d(f^n(x), f^n(y)) > \varepsilon$;
- *strongly sensitive*, if there exists $\varepsilon > 0$ such that for all $x \in Z$ and open neighbourhood V of x there exist $n_0 \geq 0$ and $y \in V$ with $d(f^n(x), f^n(y)) > \varepsilon$ for all $n \geq n_0$.

As it has been mentioned in [7, Theorem 3], (Z, f) is sensitive if and only if there exists $\varepsilon > 0$ such that $\{(x, y) \in Z \times Z : \limsup\limits_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon\}$ is dense in $Z \times Z$. Using a similar method described in Note 2.1 being sensitive (resp. strongly sensitive) does not depend on compatible metric on Z. Now we are ready to prove that the system $(X^\Gamma, \sigma_\varphi)$ is sensitive (resp. strongly sensitive) if and only if it is Li-Yorke chaotic, i.e. $W(\varphi) \neq \varnothing$.

Theorem 4.1. *If $W(\varphi) \neq \varnothing$, then $(X^\Gamma, \sigma_\varphi)$ is strongly sensitive.*

Proof. Suppose $W(\varphi) \neq \varnothing$ and choose $\theta \in W(\varphi)$, we may suppose $\theta = \beta_k$ (see Convention 2.2). If U is an open neighbourhood of $x = (x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$, there exists finite subset F of Γ such that for $\{(y_\alpha)_{\alpha\in\Gamma} \in X^\Gamma : \forall \alpha \in F\ y_\alpha = x_\alpha\} \subseteq U$. Since $\{\varphi^n(\theta)\}_{n\geq 1}$ is a one to one sequence, there exists $N \geq 1$ such that for all $n \geq N$ we have $\varphi^n(\theta) \notin F$, i.e. $\{\varphi^n(\theta) : n \geq N\} \subseteq \Gamma \setminus F$. So

$$\{(y_\alpha)_{\alpha\in\Gamma} \in X^\Gamma : \forall \alpha \neq \varphi^N(\theta), \varphi^{N+1}(\theta), \varphi^{N+2}(\theta), \ldots,\ y_\alpha = x_\alpha\} \subseteq U\,.$$

For all $m \geq N$, choose $p_m \in X \setminus \{x_{\varphi^m(\theta)}\}$, also let

$$z_\alpha = \begin{cases} p_m & \alpha = \varphi^m(\theta), m \geq N \,, \\ x_\alpha & \text{otherwise} \,. \end{cases}$$

Then $(z_\alpha)_{\alpha\in\Gamma} \in U$ and for $m \geq N$, $(u_\alpha)_{\alpha\in\Gamma} := \sigma_\varphi^m((z_\alpha)_{\alpha\in\Gamma})$, $(v_\alpha)_{\alpha\in\Gamma} := \sigma_\varphi^m((x_\alpha)_{\alpha\in\Gamma})$ we have

$$\begin{aligned} D(\sigma_\varphi^m((x_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^m((z_\alpha)_{\alpha\in\Gamma})) &= D((v_\alpha)_{\alpha\in\Gamma}, (u_\alpha)_{\alpha\in\Gamma}) \\ &\geq \frac{\delta(v_\theta, u_\theta)}{2^k} = \frac{\delta(x_{\varphi^m(\theta)}, z_{\varphi^m(\theta)})}{2^k} \\ &= \frac{\delta(x_{\varphi^m(\theta)}, p_m)}{2^k} = \frac{1}{2^k} \,. \end{aligned}$$

Hence, $(X^\Gamma, \sigma_\varphi)$ is strongly sensitive. □

Theorem 4.2. *If $W(\varphi) = \varnothing$, then $(X^\Gamma, \sigma_\varphi)$ is not sensitive.*

Proof. Suppose $W(\varphi) = \varnothing$ and consider arbitrary $\varepsilon > 0$, then there exists $N \geq 1$ such that $\frac{1}{2^N} < \varepsilon$. Since $W(\varphi) = \varnothing$, for all $\alpha \in \Gamma$ the set $\{\varphi^n(\alpha) : n \geq 0\}$ is finite. Thus

$$\Lambda := \{\varphi^n(\beta_i) : i \in \{1, \ldots, N\}, n \geq 0\}$$

is finite too, for $x = (x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$, $U = \{(y_\alpha)_{\alpha\in\Gamma} \in X^\Gamma : \forall\alpha \in \Lambda\ (y_\alpha = x_\alpha)\}$ is an open neighbourhood of $(x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$. For $n \geq 0$ and $(y_\alpha)_{\alpha\in\Gamma} \in U$ let $(v_\alpha)_{\alpha\in\Gamma} := \sigma_\varphi^n((x_\alpha)_{\alpha\in\Gamma})$ and $(w_\alpha)_{\alpha\in\Gamma} := \sigma_\varphi^n((y_\alpha)_{\alpha\in\Gamma})$, then for all $\alpha \in \Lambda$ we have $\varphi^n(\alpha) \in \Lambda$ and $v_\alpha = x_{\varphi^n(\alpha)} = y_{\varphi^n(\alpha)} = w_\alpha$, thus

$$D(\sigma_\varphi^n((x_\alpha)_{\alpha\in\Gamma}), \sigma_\varphi^n((y_\alpha)_{\alpha\in\Gamma})) = \sum_{i\geq 1, \beta_i \notin \Lambda} \frac{\delta(v_{\beta_i}, w_{\beta_i})}{2^i} \leq \sum_{i>N} \frac{1}{2^i} = \frac{1}{2^N} < \varepsilon \,.$$

So for all $\varepsilon > 0$ and $x \in X^\Gamma$ there exists open neighbourhood U of x such that $D(\sigma_\varphi^n(x), \sigma_\varphi^n(y)) < \varepsilon$ for all $y \in U$ and $n \geq 0$, which leads to the desired result. □

By using Lemmas 3.3 and 3.2, we have the following theorem.

Theorem 4.3. *If $\varphi : \Gamma \to \Gamma$ does not have any periodic point, then $(X^\Gamma, \sigma_\varphi)$ is Li-Yorke sensitive.*

Proof. Suppose $\varphi : \Gamma \to \Gamma$ does not have any periodic point then by Lemma 3.3 there exists $\mu > 0$ such that for all $a \in X^\Gamma$, there exists $b_a \in X^\Gamma$ with

$$(a, b_a) \in S(X^\Gamma, \sigma_\varphi) \wedge \limsup_{t\to\infty} D(\sigma_\varphi^t(a), \sigma_\varphi^t(b_a)) = \mu \,.$$

Consider $x = (x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$, and $b_x =: y = (y_\alpha)_{\alpha\in\Gamma}$. For all $n \geq 1$ define $y_n = (y_\alpha^n)_{\alpha\in\Gamma}$ with:

$$y_\alpha^n = \begin{cases} x_\alpha & \alpha \in \{\beta_1, \ldots, \beta_n\} \,, \\ y_\alpha & \text{otherwise} \,. \end{cases}$$

Using Lemma 3.2 for all $n \geq 1$ we have $\limsup_{t\to\infty} D(\sigma_\varphi^t(x), \sigma_\varphi^t(y_n)) = \limsup_{t\to\infty} D(\sigma_\varphi^t(x), \sigma_\varphi^t(b_x)) = \mu$ and $\liminf_{t\to\infty} D(\sigma_\varphi^t(x), \sigma_\varphi^t(y_n)) = \liminf_{t\to\infty} D(\sigma_\varphi^t(x), \sigma_\varphi^t(b_x)) = 0$, in particular $(x, y_n) \in S_{\frac{\mu}{2}}(X^\Gamma, \sigma_\varphi)$. Moreover by $\lim_{n\to\infty} y_n = x$ for all open neighbourhood U of x there exists $n \in \mathbb{N}$ with $y_n \in U$ which completes the proof. □

Theorem 4.4. *The generalized shift dynamical system* $(X^\Gamma, \sigma_\varphi)$ *is Li–Yorke sensitive if and only if* $\varphi : \Gamma \to \Gamma$ *has at least one non–quasi–periodic point.*

Proof. If $(X^\Gamma, \sigma_\varphi)$ is Li–Yorke sensitive, then $S(X^\Gamma, \sigma_\varphi) \neq \varnothing$ and by Remark 3.1 φ has a non–quasi–periodic point.

On the other hand, if φ has a non–quasi–periodic point $\theta \in \Gamma$, then for $\Lambda = \bigcup\{\varphi^n(\theta) : n \in \mathbb{Z}\}$, $\varphi\restriction_\Lambda : \Lambda \to \Lambda$ does not have any periodic point and by Theorem 4.3 $(X^\Lambda, \sigma_{\varphi\restriction_\Lambda})$ is Li–Yorke sensitive. We may suppose $\Lambda = \{\beta_{s_1}, \beta_{s_2}, \dots\}$ with $s_1 < s_2 < \cdots$, consider $p \in X$ and equip X^Λ with metric $D_\Lambda(x, y) = D(x^*, y^*)$, where for $x = (x_\alpha)_{\alpha\in\Lambda} \in X^\Lambda$ we have $x^*_\alpha = x_\alpha$ for $\alpha \in \Lambda$, $x^*_\alpha = p$ for $\alpha \notin \Lambda$, and $x^* = (x^*_\alpha)_{\alpha\in\Gamma}$.

Since $(X^\Lambda, \sigma_{\varphi\restriction_\Lambda})$ is Li–Yorke sensitive there exists $\kappa > 0$ such that for all $x \in X^\Lambda$ and $\varepsilon > 0$ there exists $y \in X^\Lambda$ with $D_\Lambda(x, y) < \varepsilon$ and $(x, y) \in S_\kappa(X^\Lambda, \sigma_{\varphi\restriction_\Lambda})$.

Consider $z = (z_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$ and open neighbourhood V_0 of z. There exists $r > 0$ with $V := \{a \in X^\Gamma : D(a, z) < r\} \subseteq V_0$, Natural projection map $p_\Lambda : \underset{(x_\alpha)_{\alpha\in\Gamma}\mapsto(x_\alpha)_{\alpha\in\Lambda}}{X^\Gamma \to X^\Lambda}$ is open and continuous hence there exists $\varepsilon > 0$ such that

$$\{a \in X^\Lambda : D_\Lambda(p_\Lambda(z), a) < \varepsilon\} \subseteq p_\Lambda(V)\,,$$

so there exists $w \in \{a \in X^\Lambda : D_\Lambda(p_\Lambda(z), a) < \varepsilon\}$ and $y \in V$ with $p_\Lambda(y) = w$ and $(p_\Lambda(z), w) \in S_\kappa(X^\Lambda, \sigma_{\varphi\restriction_\Lambda})$, i.e. $(p_\Lambda(z), p_\Lambda(y)) \in S_\kappa(X^\Lambda, \sigma_{\varphi\restriction_\Lambda})$. For $h = (h_\alpha)_{\alpha\in\Lambda}$ let:

$$\overline{h}_\alpha := \begin{cases} h_\alpha & \alpha \in \Lambda\,, \\ z_\alpha & \text{otherwise}\,, \end{cases}$$

and $\overline{h} = (\overline{h}_\alpha)_{\alpha\in\Gamma}$. For $t \geq 0$ we have $D(\sigma^t_\varphi(z), \sigma^t_\varphi(\overline{p_\Lambda(y)})) = D_\Lambda(\sigma^t_{\varphi\restriction_\Lambda}(p_\Lambda(z)), \sigma^t_{\varphi\restriction_\Lambda}(p_\Lambda(y))$, therefore $(z, \overline{p_\Lambda(y)}) \in S_\kappa(X^\Gamma, \sigma_\varphi)$. By $D(z, \overline{p_\Lambda(y)}) \leq D(z, y) < r$ we have $\overline{p_\Lambda(y)} \in V \subseteq V_0$ and obtain the desired result. □

Then, by using Theorems 4.1, 4.2 and 4.4 we have the following theorem:

Theorem 4.5 (sensitive generalized shifts)**.** *The following statements are equivalent:*

1. $(X^\Gamma, \sigma_\varphi)$ *is strongly sensitive;*
2. $(X^\Gamma, \sigma_\varphi)$ *is sensitive;*
3. $(X^\Gamma, \sigma_\varphi)$ *is Li–Yorke sensitive;*
4. $\varphi : \Gamma \to \Gamma$ *has at least one non–quasi–periodic point.*

Proof. It's clear that (1) implies (2). By Theorem 4.2, (2) implies (4). By Theorem 4.1, (4) implies (1). By Theorem 4.4, (3) and (4) are equivalent. □

4.1 Two Diagrams

By [18, Theorem 2.13], the system $(X^\Gamma, \sigma_\varphi)$ is Devaney chaotic (resp. topological transitive) if and only if $\varphi : \Gamma \to \Gamma$ is one to one without periodic points.

Now we are ready to summarize Sections 3 and 4 in the following diagram, which completes the mentioned diagram in the Introduction:

$$\begin{array}{ccccc} \varphi \text{ is one to one and } Per(\varphi)=\varnothing & \Longrightarrow & W(\varphi)=\Gamma & \Longrightarrow & W(\varphi)\neq\varnothing \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ (X^\Gamma,\sigma_\varphi) \text{ is Devaney chaotic} & & (X^\Gamma,\sigma_\varphi) \text{ is densely chaotic} & & (X^\Gamma,\sigma_\varphi) \text{ is Li}-\text{Yorke chaotic} \end{array}$$

Let:

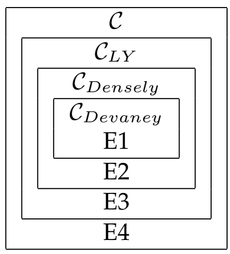
$\mathcal{C} := \{(X^\Gamma, \sigma_\eta) : \eta \in \Gamma^\Gamma\}$,

$\mathcal{C}_{Devaney} := \{(X^\Gamma, \sigma_\eta) \in \mathcal{C} : (X^\Gamma, \sigma_\eta)$ is Devaney chaotic$\}$,

$\mathcal{C}_{Densely} := \{(X^\Gamma, \sigma_\eta) \in \mathcal{C} : (X^\Gamma, \sigma_\eta)$ is Densely chaotic$\}$,

$\mathcal{C}_{LY} := \{(X^\Gamma, \sigma_\eta) \in \mathcal{C} : (X^\Gamma, \sigma_\eta)$ is Li–Yorke chaotic$\}$.

For infinite Γ, suppose β_ns are distinct, then we have the following diagram:



Where "Ei" denotes Example $(X^\Gamma, \sigma_{\varphi_i})$ for $\varphi_i : \Gamma \to \Gamma$ with:

- $\varphi_1(\beta_n) = \beta_{2n}$ for $n \geq 1$,
- $\varphi_2(\beta_1) = \varphi_2(\beta_2) = \beta_3$, and $\varphi_2(\beta_n) = \beta_{2n}$ for $n \geq 3$,
- $\varphi_3(\beta_1) = \varphi_3(\beta_2) = \beta_2$, and $\varphi_3(\beta_n) = \beta_{2n}$ for $n \geq 3$,
- $\varphi_4(\beta_1) = \beta_1$, and $\varphi_4(\beta_n) = \beta_{n-1}$ for $n \geq 2$.

4.2 On More Types of Sensitivity

This subsection considers dynamical system (Z, f) with compact metric phase space (Z, d).

Note 4.6. According to [14], the dynamical system (Z, f) is asymptotic sensitive if there exists $\varepsilon > 0$ such that for all $x \in Z$ and open neighbourhood V of x there exists $y \in V$ with $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon$. Let's verify the following diagram:

$$\text{strongly sensitive} \Rightarrow \text{asymptotic sensitive} \Rightarrow \text{sensitive.}$$

Note that if (Z, f) is strongly sensitive, then there exists $\kappa > 0$ such that for all $x \in Z$ and open neighbourhood V of x there exist $n_0 \geq 0$ and $y \in V$ with $d(f^n(x), f^n(y)) > \kappa$ for all $n \geq n_0$, hence $\limsup_{n\to\infty} d(f^n(x), f^n(y)) = \limsup_{\substack{n\to\infty\\ n\geq n_0}} d(f^n(x), f^n(y)) \geq \kappa > \frac{\kappa}{2} =: \mu$, thus (Z, f) is asymptotic sensitive.

Moreover if (Z, f) is asymptotic sensitive, then there exists $\mu > 0$ such that for all $x \in Z$ and open neighbourhood V of x there exists $y \in V$ with $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \mu$, thus $\{n \geq 1 : d(f^n(x), f^n(y)) > \mu\}$ is infinite and in particular it is nonempty. Hence (Z, f) is sensitive.

For $\varepsilon > 0$ and open subset V of Z let $N(V, \varepsilon) := \{n \geq 0 : \exists x, y \in V\ (d(f^n(x), f^n(y)) > \varepsilon)\}$ and we call $A \subseteq \mathbb{N} \cup \{0\}$ *syndetic* if there exists $N \geq 1$ with $\{i, i+1, \dots, i+N\} \cap A \neq \varnothing$ for all $i \geq 1$.

Note 4.7. (Z, f) is sensitive if and only if there exists $\kappa > 0$ such that for all opene subset V of Z, $N(V, \kappa) \neq \varnothing$.

First suppose (Z, f) is sensitive, then there exists $\varepsilon > 0$ such that for all $x \in X$ and open neighbourhood V of x there exists $y \in V$ and $n \geq 0$ with $d(f^n(x), f^n(y)) > \varepsilon$. For opene subset W of Z choose $z_1 \in W$, then there exists $z_2 \in W$ and $m \geq 0$ with $d(f^m(z_1), f^m(z_2)) > \varepsilon$, thus $m \in N(W, \varepsilon)$ and $N(W, \varepsilon) \neq \varnothing$ for all opene subset W of Z.

Now suppose there exists $\mu > 0$ such that $N(V, \kappa) \neq \varnothing$ for all opene subset V of Z. For all $x \in Z$ and open neighbouhhood V of x we have $N(V, \kappa) \neq \varnothing$ thus there exist $y, z \in V$ and $n \geq 0$ with $d(f^n(z), f^n(y)) > \kappa$, therefore $d(f^n(x), f^n(y)) > \frac{\kappa}{2}$ or $d(f^n(x), f^n(z)) > \frac{\kappa}{2}$. Thus (Z, f) is sensitive.

Note 4.8. According to [8] we call (Z, f) *syndetically sensitive* (resp. *cofinitely sensitive*) if there exists $\varepsilon > 0$ such that for all opene subset V of Z, $N(V, \varepsilon)$ is syndetic (resp. cofinite). Let's verify the following diagram:

strongly sensitive $\Rightarrow$ cofinitely sensitive $\Rightarrow$ syndetically sensitive $\Rightarrow$ sensitive.

Note that if (Z, f) is strongly sensitive, then there exists $\kappa > 0$ such that for all $x \in Z$ and open neighbourhood V of x there exist $n_0 \geq 0$ and $y \in V$ with $d(f^n(x), f^n(y)) > \kappa$ for all $n \geq n_0$. Thus for opene W of Z choose $z \in W$, there exist $m \geq 0$ and $y \in W$ with $d(f^n(z), f^n(y)) > \kappa$ for all $n \geq m$, thus $\{m, m+1, \dots\} \subseteq \{n \geq 0 : \exists p, q \in W\ (d(f^n(p), f^n(q)) > \kappa)\} = N(W, \kappa)$ and $N(W, \kappa)$ is cofinite, Hence (Z, f) is cofinitely sensitive.

Since any cofinite subset of $\mathbb{N} \cup \{0\}$ is syndetic, if (Z, f) is cofinitely sensitive, then it is syndetically sensitive. Since any syndetic subset of $\mathbb{N} \cup \{0\}$ is nonempty, if (Z, f) is syndetically sensitive, then it is sensitive (use Note 4.7).

Note 4.9. Regarding [8] we call (Z, f) *multi–sensitive* if there exists $\varepsilon > 0$ such that for all $k \geq 1$ and opene subsets $V_1, \dots, V_k$ of Z, we have $\bigcap_{1\leq n\leq k} N(V_n, \varepsilon) \neq \varnothing$. Let's verify the following diagram:

strongly sensitive $\Rightarrow$ multi-sensitive $\Rightarrow$ sensitive.

Note that if (Z, f) is strongly sensitive, then there exists $\kappa > 0$ such that for all $x \in Z$ and open neighbourhood V of x there exist $n_0 \geq 0$ and $y \in V$ with $d(f^n(x), f^n(y)) > \kappa$ for all $n \geq n_0$.

Suppose $V_1 \ldots, V_k$ are opene subsets of Z, for each i choose $x_i \in V_i$, then there exists $n_i \geq 0$ and $y_i \in V_i$ with $d(f^n(x_i), f^n(y_i)) > \kappa$ for all $n \geq n_i$. Thus for $m = \max(n_1, \cdots, n_k)$ we have $\{m, m+1, \ldots\} \subseteq \bigcap\limits_{1 \leq i \leq k} N(V_i, \kappa)$, in particular $\bigcap\limits_{1 \leq i \leq k} N(V_i, \kappa) \neq \varnothing$ and $(Z, f)$is multi–sensitive.

By Note 4.7, if (Z, f) is multi–sensitive, then it is sensitive.

Lemma 4.1. *If (Z, f) is cofinitely sensitive, then there exists $\varepsilon > 0$ such that*

$$\lim_{n \to \infty} \frac{|N(V, \varepsilon) \cap \{0, \ldots, n\}|}{n+1} = 1$$

for all opene V subset of Z.

Proof. Since (Z, f) is cofinitely sensitive there exists $\varepsilon > 0$ such that for all opene subset V of Z, $N(V, \varepsilon)$ is cofinite. Hence for opene subset V of Z there exists $m \geq 1$ such that $\{m, m+1, \ldots\} \subseteq N(V, \varepsilon)$, thus for all $n \geq m$ we have

$$\frac{n-m+1}{n+1} = \frac{|\{m, m+1, \ldots n\} \cap \{0, \ldots, n\}|}{n+1} \leq \frac{|N(V, \varepsilon) \cap \{0, \ldots, n\}|}{n+1} \leq 1$$

which leads to $\lim\limits_{n \to \infty} \frac{|N(V, \varepsilon) \cap \{0, \ldots, n\}|}{n+1} = 1$ and completes the proof. □

Note 4.10. We call (Z, f) *ergodically sensitive* [8] if there exists $\varepsilon > 0$ such that for all opene subset V of Z, $\limsup\limits_{n \to \infty} \frac{|N(V, \varepsilon) \cap \{0, \ldots, n\}|}{n+1} > 0$. By Notes 4.8, 4.7 and Lemma 4.1 it is easy to see that we have the following diagram:

strongly sensitive ⇒ cofinitely sensitive ⇒ ergodically sensitive ⇒ sensitive.

Theorem 4.11. *For finite discrete X with at least two elements, nonempty countable set Γ and $\varphi : \Gamma \to \Gamma$ the following statements are equivalent:*

1. *the system $(X^\Gamma, \sigma_\varphi)$ is Li-Yorke chaotic (i.e. $(X^\Gamma, \sigma_\varphi)$ has an scrambled pair by [17, Theorem 3.3]);*
2. *the system $(X^\Gamma, \sigma_\varphi)$ is topological chaotic;*
3. *the system $(X^\Gamma, \sigma_\varphi)$ is spatiotemporally chaotic;*
4. *the system $(X^\Gamma, \sigma_\varphi)$ is sensitive;*
5. *the system $(X^\Gamma, \sigma_\varphi)$ is Li–Yorke sensitive;*
6. *the system $(X^\Gamma, \sigma_\varphi)$ is strongly sensitive;*
7. *the system $(X^\Gamma, \sigma_\varphi)$ is asymptotic sensitive;*
8. *the system $(X^\Gamma, \sigma_\varphi)$ is syndetically sensitive;*
9. *the system $(X^\Gamma, \sigma_\varphi)$ is cofinitely sensitive;*
10. *the system $(X^\Gamma, \sigma_\varphi)$ is multi-sensitive;*
11. *the system $(X^\Gamma, \sigma_\varphi)$ is ergodically sensitive;*

12. *the map $\varphi : \Gamma \to \Gamma$ has at least non–quasi–periodic point.*

Proof. (1), (2), and (12) are equivalent by Remark 3.1.
(4), (5), (6), and (12) are equivalent by Theorem 4.5.
(4), and (7) are equivalent by Theorem 4.5 and Note 4.6.
(4), (8), and (9) are equivalent by Theorem 4.5 and Note 4.8.
(4), and (10) are equivalent by Theorem 4.5 and Note 4.9.
(4), and (11) are equivalent by Theorem 4.5 and Note 4.10.
So (1) and (5) are equivalent.

In order to complete the proof, it's enough to show (5) imply (3), and (3) imply (1).

(5⇒3): Suppose $(X^\Gamma, \sigma_\varphi)$ is Li–Yorke sensitive. Then there exists $\kappa > 0$ such that for every $x \in X^\Gamma$ and open neighbourhood U of x there exists $y \in U$ with $(x, y) \in S_\kappa(X^\Gamma, \sigma_\varphi)$, since $S_\kappa(X^\Gamma, \sigma_\varphi) \subseteq S(X^\Gamma, \sigma_\varphi)$, we have $(x, y) \in S(X^\Gamma, \sigma_\varphi)$ and x, y are scrambled, hence $(X^\Gamma, \sigma_\varphi)$ is spatiotemporally chaotic.

(3⇒1): Suppose $(X^\Gamma, \sigma_\varphi)$ is spatiotemporally chaotic. Then for every $x \in X^\Gamma$ and open neighbourhood U of x there exists $y \in U$ such that x, y are scrambled. Choose $a \in X^\Gamma$, then there exists $b \in X^\Gamma$ such that a, b are scrambled and (1) is valid. □

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